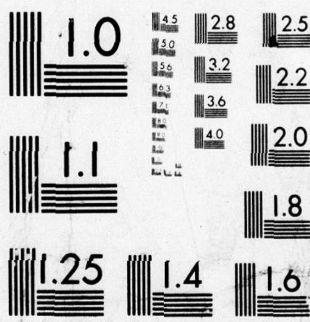
AD-A041 667 NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CONN NEW --ETC F/G 12/1
AN ADDITION FORMULA FOR GREEN'S FUNCTIONS.(U)
JUN 77 J S COHEN, J S PAPADAKIS

UNCLASSIFIED NUSC-TR-5693 NL

| OF |
AD
A041667

END
DATE
FILMED
8-77



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NUSC Technical Report 5693

NUSC Technical Report 5693

12

NW

An Addition Formula for Green's Functions

Jeffrey S. Cohen
John S. Papadakis
Special Projects Department

14 June 1977

NUSC
NAVAL UNDERWATER SYSTEMS CENTER
Newport, Rhode Island • New London, Connecticut

PREFACE

This work was performed under NUSC overhead funds. The authors are grateful to the Naval Underwater Systems Center and particularly to L. T. Einstein (Code 314) for supporting this study.

REVIEWED AND APPROVED: 14 June 1977

R. W. Hasse
Head: Special Projects Department

NOTE

This study was submitted by Jeffrey S. Cohen and accepted May 1977 by the University of Rhode Island as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

The authors of this report are located at the New London Laboratory, Naval Underwater Systems Center, New London, Connecticut 06320.

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NUSC TR-5693	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER Technical rept.	
4. TITLE (and Subtitle) AN ADDITION FORMULA FOR GREEN'S FUNCTIONS.	5. TYPE OF REPORT & PERIOD COVERED
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jeffrey S. Cohen John S. Papadakis	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Underwater Systems Center New London Laboratory New London, CT 06320	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS A91475
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Underwater Systems Center Newport Laboratory Newport, RI	12. REPORT DATE 14 June 1977
	13. NUMBER OF PAGES 62
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 65p.	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Green's Functions
Elliptic Partial Differential Equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A formula yielding an expression for the Green's function of a linear separable elliptic partial differential equation in terms of the Green's function for two simpler equations, one of which is elliptic and the other hyperbolic, is derived utilizing separation of variables, transform techniques for solving ordinary differential equations, Parseval's identity from Fourier transform theory, and general properties of Green's and Riemann functions. The formula is applied successfully to specific problems, some

OVER

DD FORM 1 JAN 73 1473

405 918

20. (Cont'd)

of which are believed to be previously unsolved, and Green's functions are represented using the special functions of mathematical physics. A more detailed overview of the thesis may be found in Appendix A, which is the recommended starting point for reading this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

FOREWORD

In this thesis, an addition formula for Green's functions of linear separable elliptic partial differential equations in two independent variables is derived. The formula is applied to generate new Green's functions from known Green's functions.

This thesis is in manuscript format: it consists of one manuscript and six supporting appendices. The recommended starting point for reading the thesis is Appendix A: Overview of the Thesis.

AN ADDITION FORMULA FOR GREEN'S FUNCTIONS

by

JEFFREY S. COHEN and JOHN S. PAPADAKIS

Naval Underwater Systems Center
New London Laboratory
New London, Connecticut 06320

1. INTRODUCTION

One of the most general approaches to solving boundary value problems for linear differential equations consists of expressing the solution in terms of an auxiliary function called the Green's function. The Green's function is usually defined to be the solution of the adjoint problem with a delta function as a nonhomogeneous term. The particular boundary conditions affect the formulation of the adjoint problem and thus, the Green's function as well. The Green's function method can be applied to elliptic, parabolic, and hyperbolic linear partial differential equations as well as linear ordinary differential equations.

This paper presents a derivation of a formula that gives a form of the Green's function of a separable elliptic partial differential equation in terms of the Green's functions for two simpler equations, one of which is elliptic and the other hyperbolic. Because of the presence of the hyperbolic equation and for the sake of completeness, we shall, in section 2, briefly review the definition of the Green's function for a Cauchy problem and its relationship to the Riemann function. Then, the Riemann function, which is known in some special cases, can be used to find the Green's function in these cases. In section 3, the formula will be formally derived, applied to the reduced wave equation, and be further justified. In section 4, we shall apply the formula to derive what we believe to be a new Green's function.

Before proceeding however, it is interesting to note that although we are primarily interested in Green's functions for elliptic problems, the motivation for this work came from recent papers on hyperbolic problems. A second-order linear hyperbolic equation in two independent variables may be solved using the Riemann function. In 1958, Copson [Ref. 1] formally derived a technique giving a possible form for the Riemann function (which Copson called the Riemann-Green function), assuming the variables are separable. The technique utilized integral transforms which generalized Riemann's own use of Fourier cosine transforms. In 1964, Mackie [Ref. 2] investigated similar problems and presented a close relationship between the Riemann function and the Green's function for the Cauchy problem. Mackie then applied Fourier-Bessel transforms to find the Green's function for Riemann's original example and utilized the relationship between Riemann and Green's functions to find the Riemann functions. Recently, Papadakis and Wood [Ref. 3] developed an addition formula that gives the Riemann function of a separable equation in terms of the Riemann functions of two simpler equations. In deriving the formula, they employed a refinement of Copson's integral transform method to find Green's functions and then utilized Mackie's relationship to find the Riemann function. In this paper, the technique is further developed by applying it to Green's functions for elliptic partial differential equations.

2. PRELIMINARIES

a. The Green's Function for Elliptic Equations

Consider the linear elliptic second-order partial differential equation in two variables

$$E[u] = u_{xx} + u_{yy} + 2a(x,y)u_x - 2b(x,y)u_y + c(x,y)u(x,y) = f(x,y) \tag{2.1}$$

with suitable boundary conditions on a closed curve Γ (e.g., $u = 0$ on Γ). The Green's function for (2.1), $G(x,y;X,Y)$ is defined as the solution to the adjoint equation

$$E^*[G] = G_{xx} + G_{yy} - 2(aG)_x + 2(bG)_y + cG = \delta(x-X)\,\delta(y-Y) \tag{2.2}$$

with suitable (adjoint) conditions on the boundary Γ (e.g., $G = 0$ on Γ). If D is the interior of Γ, the solution of (2.1) can be written in the form

$$u(X,Y) = \int_D\int G(x,y;X,Y)f(x,y)\,dxdy + \int_\Gamma B[u,G] \tag{2.3}$$

where $B[u,G]$ is known on Γ (e.g., $B[u,G] = 0$ on Γ).

b. The Green's function for Hyperbolic Equations

The concept of the Green's function is also meaningful for the Cauchy problem

$$H[u] = u_{xx} - u_{yy} + 2a(x,y)u_x - 2b(x,y)u_y$$
$$+ c(x,y)u(x,y) = f(x,y) \qquad (2.4)$$

with initial value u and u_y given on the line $y = y_0$. As in the elliptic case, the Green's function $G(x,y;X,Y)$ is the solution of

$$H^*[G] = G_{xx} - G_{yy} - 2(aG)_x + 2(bG)_y + cG$$
$$= \delta(x-X)\ \delta(y-Y) \qquad (2.5)$$

subject to the adjoint boundary conditions

$$G = \frac{\partial G}{\partial n} = 0 \quad \text{on} \quad \Gamma \qquad (2.6)$$

where $\frac{\partial G}{\partial n}$ is the (outward) normal derivative of G on Γ, and Γ is a non-characteristic curve (i.e., the absolute value of the slope of Γ is always less than 1) such that Γ and a segment, $\mathcal{L}$, of the line $y = y_0$ form a closed curve, as in Fig. 1. From (2.5), (2.6), and the amount of freedom we have in choosing Γ, it can be seen that $G = 0$ in the portion of the interior of $\Gamma+\mathcal{L}$ outside the triangle Δ, with vertices at points P, A, and B. Now the solution u of (2.4) is given by

$$u(X,Y) = \iint_\Delta G(x,y;X,Y)f(x,y)\ dxdy$$
$$- \int_{\mathcal{L}} (u_yG-uG_y+2buG)\ dx. \qquad (2.7)$$

c. The Riemann Function

For the Cauchy problem described above, the Riemann function $R(x,y;X,Y)$ is the solution of

$$H^*[R] = 0 \tag{2.8a}$$

satisfying

$$R_x + R_y = (a+b)R \quad \text{on} \quad y-Y = x-X \tag{2.8b}$$

$$R_x - R_y = (a-b)R \quad \text{on} \quad y-Y = -(x-X) \tag{2.8c}$$

$$R(X,Y;X,Y) = 1. \tag{2.8d}$$

Now the solution of (2.4) is given by

$$\begin{aligned} u(X,Y) = &-\frac{1}{2}\iint_{\Delta} R(x,y;X,Y)f(x,y)\,dxdy \\ &+ \frac{1}{2}\int_A^B (u_y R - uR_y + 2buR)\,dx \\ &+ \frac{1}{2}[u(A)R(A;P) + u(B)R(B;P)]. \end{aligned} \tag{2.9}$$

If the initial values of u and u_y are homogeneous (i.e., identically 0) along $y = y_0$, a comparison of (2.7) and (2.9) would lead to

$$G(x,y;X,Y) = -\frac{1}{2}R(x,y;X,Y) \tag{2.10}$$

inside the triangle Δ. If the initial data along $y = y_0$ is nonhomogeneous, the contour integral in (2.7) requires integration of terms including G and G_y along lines across which G itself is discontinuous. This generates terms analogous to those inside the brackets in (2.9). Thus, (2.10) holds for arbitrary initial data. (Except for a difference in notation between a Green's function and an adjoint Green's function, the analysis leading to (2.10) is due to Mackie [Ref. 2].)

3. THE ADDITION FORMULA

a. Formal Derivation

Let us consider the separable elliptic equation

$$L[u] = u_{xx} + u_{yy} + [c_1(x)+c_2(y)]u(x,y) = f(x,y) \qquad (3.1)$$

and attempt to find the "free-space" Green's function (i.e., assuming $c_1(x)$ and $c_2(y)$ are analytic, find a Green's function over the entire (x,y) plane). Since L is self-adjoint, $G(x,y;X,Y)$ must satisfy

$$L^*[G] \equiv L[G] = \delta(x-X)\ \delta(y-Y). \qquad (3.2)$$

We shall attempt to find an expression for G in terms of the Green's function G_1 satisfying the hyperbolic equation

$$G_{1_{xx}} - G_{1_{yy}} + c_1(x)G_1 = \delta(x-X)\ \delta(y-Y) \qquad (3.3)$$

and the Green's function G_2 satisfying the elliptic equation

$$G_{2_{xx}} + G_{2_{yy}} + c_2(y)G_2 = \delta(x-X)\ \delta(y-Y). \qquad (3.4)$$

It has been shown by Papadakis and Wood [Ref. 3] that

$$R_1(x,y;X,Y) \equiv R_1(x,y-Y;X,0) \equiv R_1(x,Y-y;X,0). \qquad (3.5)$$

Recalling (2.10) we can say that G_1 is also an even function of $y-Y$ wherever it is defined for both $+(y-Y)$ and $-(y-Y)$. We should like to make a similar claim for G_2. If we assume a condition at ∞, call it a "radiation condition," ensuring the uniqueness of G_2, then we can claim that

$$G_2(x,y;X,Y) \equiv G_2(x-X,y;0,Y) \equiv G_2(X-x,y;0,Y). \tag{3.6}$$

Clearly, any form of (3.6) satisfies (3.4). Thus, if our "radiation condition" is also satisfied by any form of (3.6), the claim is justified by the uniqueness of G_2. Since we are proceeding formally at this stage, let us accept these hypotheses and (3.6) as established, with this final observation. If $c_2(y) \equiv k^2$ (the case with which we will be primarily concerned in this paper) and the well-known Sommerfeld radiation condition [Ref. 4] is assumed, then G_2 is unique and an even function of $x-X$ as desired.

Now, let us proceed by separating variables in the homogeneous form of (3.2), leading to a pair of ordinary differential equations, one of which is

$$\theta''(x) + [c_1(x) + \lambda^2]\theta(x) = 0 \tag{3.7}$$

where λ^2 is the separation constant. Assume there exists a solution $\theta(x,\lambda)$ of (3.7) that defines a transform

$$f(\lambda) = \int \theta(x,\lambda)F(x)\,dx.$$

Let $\hat{\theta}(x,\lambda)$ be the inverse transform so that

$$F(x) = \int \hat{\theta}(x,\lambda)f(\lambda)\,d\lambda.$$

Applying the transform to (3.2), integrating by parts twice (ignoring terms evaluated at the endpoints of the integral), and recalling that $\theta(x,\lambda)$ is a solution of (3.7) yields

$$g_{YY} + [c_2(y) - \lambda^2]g = \delta(y-Y) \tag{3.9}$$

where g is defined by

$$g(y;X,Y,\lambda)\theta(X,\lambda) \equiv \int \theta(x,\lambda)G(x,y;X,Y)\,dx.$$

Utilizing the inverse transform gives

$$G(x,y;X,Y) = \int \hat{\theta}(x,\lambda)\theta(X,\lambda)g(y;X,Y,\lambda)\,d\lambda. \tag{3.10}$$

Now let us apply the same procedure to (3.3). Separating variables leads to (3.7) again and therefore, the same solution $\theta(x,\lambda)$. Transforming (3.3) yields

$$g_{1_{YY}} + \lambda^2 g_1 = -\delta(y-Y) \tag{3.11}$$

where

$$g_1(y-Y;X,\lambda)\theta(X,\lambda) \equiv \int \theta(x,\lambda)G_1(x,y-Y;X,0)\,dx. \tag{3.12}$$

Recalling boundary condition (2.6) and Fig. 1, (3.11) can be solved uniquely with the initial conditions $g_1 \to 0$ and $g_{1_y} \to 0$ as $y \to -\infty$:

$$g_1(y-Y;X,\lambda) = -\frac{\sin\lambda(y-Y)}{\lambda} H(y-Y)$$

where H is the Heaviside step function. Inverse transforming (3.12) yields

$$G_1(x,y-Y;X,0) = -H(y-Y) \int \hat{\theta}(x,\lambda)\theta(X,\lambda)\frac{\sin\lambda(y-Y)}{\lambda}d\lambda. \quad (3.13)$$

For the last time, separating variables in (3.4) leads to the equation

$$\phi''(x) + \lambda^2\phi(x) = 0$$

with a solution of the form $e^{-i\lambda x}$, yielding a Fourier transform. As before, we transform (3.4) and find

$$g_{2_{YY}} + [c_2(y)-\lambda^2]g_2 = \delta(y-Y) \quad (3.14)$$

where

$$g_2(y;X,Y,\lambda)e^{-i\lambda X} \equiv \int_{-\infty}^{\infty} e^{-i\lambda x}G_2(x-X,y;0,Y)\,dx. \quad (3.15)$$

Recalling from (3.6) that G_2 is an even function of x-X, it is easily deduced that g_2 is an even function of λ and independent of X. Also comparing (3.14) to (3.9) leads to the conclusion that $g_2 \equiv g$ and therefore

$$g(y;X,Y,\lambda) \equiv g(y,Y;\lambda) \equiv g(y,Y;-\lambda).$$

Now, inverse transforming (3.15) and using the fact that G_2 and g are even

$$G_2(x-X,y;0,Y) = \frac{1}{\pi}\int_0^\infty \cos\lambda(x-X)g(y,Y;\lambda)\,d\lambda. \tag{3.16}$$

Let us formally assume that the integrals in (3.10) and (3.13) also have limits from 0 to ∞. If, in (3.13), we denote $y-Y$ by t and investigate for $t>0$ we get

$$G_{1_t}(x,t;X,0) = -\int_0^\infty \hat{\theta}(x,\lambda)\theta(X,\lambda)\cos\lambda t\,d\lambda. \tag{3.17}$$

Rewriting (3.16) with t replacing $x-X$

$$G_2(t,y;0,Y) = \frac{1}{\pi}\int_0^\infty g(y,Y;\lambda)\cos\lambda t\,d\lambda. \tag{3.18}$$

An application of Parseval's identity to (3.17) and (3.18) yields

$$\begin{aligned}\int_0^\infty G_{1_t}(x,t;X,0)G_2(t,y;0,Y)\,dt \\ &= -\frac{1}{2}\int_0^\infty \hat{\theta}(x,\lambda)\theta(X,\lambda)g(y,Y;\lambda)\,d\lambda \\ &= -\frac{1}{2}G(x,y;X,Y)\end{aligned}$$

the last equality coming from (3.10), with the assumed limits of integration from 0 to ∞. Thus we hope to find the Green's function of (3.1) in the form

$$G(x,y;X,Y) = -2 \int_0^\infty G_{1_t}(x,t;X,0)G_2(t,y;0,Y)\,dt. \quad (3.19)$$

Equation (3.19) is the general form that we shall use in our search for Green's functions. In order to get a more specific formula, one must investigate the behavior of G_1 and G_2 and the effects of boundaries and boundary conditions on G. For the time being, we shall continue on with our assumptions that $c_1(x)$ and $c_2(y)$ are well-behaved functions and look for a "free-space" formula. In section 4, we shall analyze (3.19) under different conditions.

Now, recalling our discussion of Green's functions and Riemann functions for hyperbolic equations, and particularly Fig. 1 and Eq. (2.10)

$$G_1(x,y;X,Y) = \begin{cases} 0 & \text{for } y-Y < |x-X| \\ -\frac{1}{2}R_1(x,y;X,Y) & \text{for } y-Y \geq |x-X| \end{cases}$$

where the Riemann function, R_1, is defined by (2.8). Therefore, we can write

$$G_1(x,t;X,0) = -\frac{1}{2} R_1(x,t;X,0)H(t-|x-X|)$$

where H is once again the Heaviside step function. Differentiating G_1 with respect to t:

$$G_{1_t}(x,t;X,0) = -\frac{1}{2} R_1(x,t;X,0)\delta(t-|x-X|)$$
$$- \frac{1}{2} R_{1_t}(x,t;X,0)H(t-|x-X|). \quad (3.20)$$

Substituting (3.20) into (3.19):

$$G(x,y;X,Y) = \int_0^\infty \{R_1\delta + R_{1_t}H\}G_2\,dt$$
$$= R_1(x,|x-X|;X,0)G_2(|x-X|,y;0,Y)$$
$$+ \int_{|x-X|}^\infty R_{1_t}(x,t;X,0)G_2(t,y;0,Y)dt\,. \quad (3.21)$$

Integrating (2.8b&c) with $a = b = 0$, as in this case, we get

$$R_1(x,|x-X|;X,0) = 1.$$

From (3.6)

$$G_2(|x-X|,y;0,Y) = G_2(x-X,y;0,Y).$$

Lastly, R_1 and G_2, as they appear in the integral in (3.21), are even functions of t so that the integrand (with its t-derivative term) is odd, and we can drop the absolute value from the lower limit of integration.

Recalling (2.10) we can now write (3.21) as

$$G(x,y;X,Y) = G_2(x-X,y;0,Y)$$
$$- 2\int_{x-X}^\infty G_{1_t}(x,t;X,0)G_2(t,y;0,Y)\,dt. \quad (3.22)$$

Integrating by parts and assuming that the term G_1G_2 evaluated at $t = \infty$ is zero, yields the alternate form

$$G(x,y;X,Y) = 2\int_{x-X}^{\infty} G_1(x,t;X,0)G_{2_t}(t,y;0,Y)\, dt. \quad (3.23)$$

Equations (3.22) and (3.23) are the relationships between the Green's functions G, G_1, and G_2 satisfying (3.2), (3.3), and (3.4) which we wished to derive from our general form (3.19). Of course, all our work so far is entirely formal and thus, justification is required. Although we do not intend to produce a rigorous proof of the validity of the formulas, we shall verify that they satisfy some of the properties of Green's functions and further justify some of our assumptions. But before continuing in this endeavor, let us convince ourselves that it is worth the effort by testing the formulas on a problem for which we know the correct answer.

b. A Test Problem - The Reduced Wave Equation

In this section, we shall utilize (3.23) to find the Green's function for (3.1) with $c_1(x) = -a^2$ and $c_2(y) = b^2$, where a and b are real, positive constants satisfying

$$k^2 = b^2 - a^2 > 0. \quad (3.24)$$

Notice that we have broken up the positive constant k^2 as the difference of two positive constants rather than the sum. We shall comment on this choice later in the section.

Now we can rewrite (3.2), (3.3), and (3.4), the equations for G, G_1, and G_2, respectively, as

$$G_{xx} + G_{yy} + k^2G = \delta(x-X)\ \delta(y-Y)\ , \tag{3.25}$$

$$G_{1_{xx}} - G_{1_{yy}} - a^2G_1 = \delta(x-X)\ \delta(y-Y), \tag{3.26}$$

and

$$G_{2_{xx}} + G_{2_{yy}} + b^2G = \delta(x-X)\ \delta(y-Y). \tag{3.27}$$

The solutions of (3.26) and (3.27) are known to be

$$G_1(x,y;X,Y) = \begin{cases} -\frac{1}{2}J_0(a\sqrt{(y-Y)^2-(x-X)^2}) & \text{for } y-Y \geq |x-X| \\ 0 & \text{for } y-Y < |x-X| \end{cases} \tag{3.28}$$

and

$$G_2(x,y;X,Y) = \frac{1}{4i}\, H_0^{(1)}(b\sqrt{(x-X)^2+(y-Y)^2}). \tag{3.29}$$

Substituting (3.28) and (3.29) into (3.23) and then performing the required differentiation yields

$$G(x,y;X,Y) = \frac{-1}{4i}\int_{x-X}^{\infty} J_0(a\sqrt{t^2-(x-X)^2})\frac{\partial}{\partial t}H_0^{(1)}(b\sqrt{t^2+(y-Y)^2})dt.$$

$$= \frac{1}{4i}\int_{x-X}^{\infty} J_0(a\sqrt{t^2-(x-X)^2})H_1^{(1)}(b\sqrt{t^2+(y-Y)^2}) \cdot$$

$$\cdot \frac{bt\ dt}{\sqrt{t^2+(y-Y)^2}} .$$

Changing the integration variable from t to

$$s = \sqrt{t^2-(x-X)^2}$$

yields

$$G(x,y;X,Y) = \frac{1}{4i}\int_0^\infty J_0(as)H_1^{(1)}(b\sqrt{s^2+r^2})\frac{bs\,ds}{\sqrt{s^2+r^2}} \tag{3.30}$$

where $r \equiv \sqrt{(x-X)^2+(y-Y)^2}$.

Utilizing (3.24) and the appropriate integral tables [Ref. 5, pg. 358, 19.4(3) and Ref. 6, pg. 706, 6.596(6)] yields

$$G(x,y;X,Y) = \frac{1}{4i} H_0^{(1)}(kr) \tag{3.31}$$

the well-known free-space Green's function for the reduced wave equation (3.25).

Returning to the remark following (3.24), we note that if k^2 were the sum of a^2 and b^2 then, in (3.28), G_1 would consist of an I_0, a modified Bessel function, rather than a J_0. If, in (3.30), the J_0 term were replaced by I_0, the integral would diverge since I_0 increases exponentially for large arguments. Thus, even under conditions when the formula is applicable, each problem must be studied with great care and a certain amount of ingenuity may be required in order to solve the problem.

Let us note that the formula also works for the case $c_1(x) = -a^2$ and $c_2(y) = -b^2$. The analysis is almost identical with the preceding example, with the elliptic Green's function $\frac{1}{4i} H_0^{(1)}$ replaced by

$\frac{-1}{2\pi} K_0$, the modified Hankel function. Reference 6 [pg. 706, 6.596(6)] provides the required integral.

c. Further Justification of the Addition Formula

In order to verify the addition formula (3.19) or formulas derived from it, such as (3.22) and (3.23), at least three properties of our representation of G should be checked:

(1) that the integrals involved converge;

(2) that as $(x,y) \to (X,Y)$, G possesses the appropriate (logarithmic) singularity; and

(3) that for $(x,y) \neq (X,Y)$, G satisfies the homogeneous equation.

Properties (1) and (2) should be verified for each specific problem investigated; that is, for the particular G_1 and G_2 that happen to arise. For formulas (3.22) and (3.23), property (3) can then be verified in general by direct differentiation.

Differentiating (3.23) twice with respect to x and recalling that for $(x,y) \neq (X,Y)$, $G_{1_{xx}} = G_{1_{yy}} - c_1 G_1$ (from (3.3))

$$G_{xx} = G_{2_{xx}} - 2G_{2_x}\left(G_{1_x}\Big|_{y-Y=x-X}\right)$$

$$+ 2\int_{x-X}^{\infty} G_{1_{tt}} G_{2_t}\, dt - c_1 G. \qquad (3.32)$$

Similarly, differentiating (3.22) twice with respect to

y and recalling that

$$G_{2_{yy}} = -G_{2_{xx}} - c_2 G_2 \quad \text{(from (3.4))}$$

$$G_{yy} = G_{2_{yy}} + 2\int_{x-X}^{\infty} G_{1_t} G_{2_{tt}} dt + 2c_2 \int_{x-X}^{\infty} G_{1_t} G_2 \, dt. \qquad (3.33)$$

Adding (3.32) and (3.33)

$$G_{xx} + G_{yy} = -c_1 G + (G_{2_{xx}} + G_{2_{yy}}) + 2c_2 \int_{x-X}^{\infty} G_{1_t} G_2 \, dt$$

$$+2\int_{x-X}^{\infty} G_{1_t} G_{2_{tt}} + G_{1_{tt}} G_{2_t}) dt - 2G_{2_x} (G_{1_x} \Big|_{y-Y=x-X})$$

$$= -c_1 G - c_2 (G_2 - 2\int_{x-X}^{\infty} G_{1_t} G_2 dt) + 2\int_{x-X}^{\infty} (G_{1_t} G_{2_t})_t dt$$

$$- 2G_{2_x} (G_{1_x} \Big|_{y-Y=x-X}).$$

So, using (3.22)

$$G_{xx} + G_{yy} + (c_1 + c_2) G$$

$$= 2(G_{1_t} G_{2_t}) \Big|_{t=x-X}^{\infty} - 2G_{2_x} (G_{1_x} \Big|_{y-Y=x-X}).$$

Recalling that we have already assumed that $G_1 G_2 \to 0$ as $t \to \infty$ in order to integrate by parts, let us further assume that $G_{1_t} G_{2_t} \to 0$ as $t \to \infty$. Then

$$G_{xx} + G_{yy} + (c_1 + c_2) G = -2G_{2_x} (G_{1_x} + G_{1_y}) \Big|_{y-Y=x-X} = 0$$

by (2.10) and integration of (2.8b, with $a = b = 0$). Thus, (3.22) and (3.23) satisfy the homogeneous equation away from the source if

$$\lim_{t\to\infty} G_1(x,t;X,0)G_2(t,y;0,Y) = 0 \tag{3.34}$$

and

$$\lim_{t\to\infty} G_{1_t}(x,t;X,0)G_{2_t}(t,y;0,Y) = 0. \tag{3.35}$$

These conditions were met by the Green's functions in our test problem.

Let us note that in some cases conditions (3.34) and (3.35) may be unnecessarily restrictive. In particular (3.34) was used to integrate (3.22) by parts and, thus, deriving (3.23). Then we used the equivalence of (3.22) and (3.23), along with (3.35), to verify that our formula did indeed yield a solution of the homogeneous differential equation. However, it is possible for (3.22) (or (3.23), for that matter) to be valid while (3.23) (or (3.22)) is not.

Consider Laplace's equation

$$u_{xx} + u_{yy} = 0. \tag{3.36}$$

With $c_1(x)$ and $c_2(y)$ both 0, G_1 and G_2 are the Green's functions for

$$u_{xx} - u_{yy} = 0 \tag{3.37}$$

and

$$u_{xx} + u_{yy} = 0 \tag{3.38}$$

respectively.

Then,

$$G_1(x,y;X,Y) = \begin{cases} -\frac{1}{2} & \text{for} \quad y-Y \geq |x-X| \\ 0 & \text{for} \quad y-Y < |x-X| \end{cases} \tag{3.39}$$

and, letting $r \equiv \sqrt{(x-X)^2+(y-Y)^2}$

$$G_2(x,y;X,Y) = \frac{1}{2\pi} \log r. \tag{3.40}$$

Substituting in (3.22) we find the well-known free-space Green's function for Laplace's equation

$$G(x,y;X,Y) = \frac{1}{2\pi} \log r \, , \tag{3.41}$$

since $G_{1_t} = 0$ inside the integral appearing in (3.22). However, (3.23) diverges for the same situation.

Now, in light of our successful general verification of property (3) and our successful applications of the formulas in the test example and Laplace's equation, let us take a heuristic look at properties (1) and (2) by placing certain restrictions on $c_1(x)$ and $c_2(y)$ and the boundary conditions associated with (3.1). For the formal derivation, we assumed that c_1 and c_2 were analytic in the entire plane. Now, let us modify this assumption so that rather than the whole plane, we are looking at an infinite domain with boundaries separable in x and y (e.g., the half-plane $x > 0$). Also let us assume that any singularities in c_1 and c_2 appear outside the domain of interest. Finally, assume that as x and y approach ∞, $c_1(x)$ and

$c_2(y)$ approach real constant values. Without loss of generality, it can be assumed that c_1 approaches a negative constant since we can subtract a constant from c_1 and add it to c_2 and not alter the total coefficient in (3.1)).

With these assumptions on c_1 and c_2, we can hope that G_1 and G_2, satisfying (3.3) and (3.4), will behave, near ∞, like J_0 and $H_0^{(1)}$ (or K_0), the Green's functions satisfying (3.26) and (3.27). From the test problem, we know that the integral (3.30) converges and thus hope that the integrals containing G_1 and G_2, (3.22) and (3.23), will also converge and that the integration by parts leading from one form to the other will be justified. If necessary, further restrictions on how fast c_1 and c_2 approach their limiting values could be made to insure asymptotic knowledge of G_1 and G_2, or perhaps entirely different types of assumptions are necessary. However, since the derivation of the addition formula has been formal, we shall leave the problem of convergence at this point.

We shall cope with the problem of the singularity as $(x,y) \to (X,Y)$ in much the same vein. Since we have assumed that c_1 and c_2 are well-behaved, we shall also assume that they can be treated locally as constants. With these assumptions we can hope that the general G_1 and G_2 will behave like their counterparts in the test problem and exhibit the appropriate singularity.

While these arguments indicate that properties (1) and (2) will be satisfied by certain types of interesting problems, let us recall that these properties should be checked for each individual problem. Property (3), however, has been directly verified under general conditions.

4. THE GREEN'S FUNCTION FOR $u_{xx} + u_{yy} + [k^2 - \frac{m(m+1)}{x^2}]u=0$

a. Derivation of the Green's Function using the Addition Formula

In this section, we shall finally apply the addition formula to a non-trivial example. To the best of our knowledge, the Green's function found in this section has not been previously presented.

Let us consider the equation

$$L[u] = u_{xx} + u_{yy} + [k^2 - \frac{m(m+1)}{x^2}]u = 0 \tag{4.1}$$

on the half-plane $x > 0$, where m is a positive integer. We shall attempt to find the Green's function $G(x,y;X,Y)$ for (4.1) on the half-plane $x > 0$ satisfying

$$L[G] = \delta(x-X)\ \delta(y-Y) \tag{4.2}$$

$$G = 0 \quad \text{on} \quad x = 0 \tag{4.2a}$$

$$G \to 0 \quad \text{as} \quad |(x,y)| \to \infty \quad . \tag{4.2b}$$

As (4.1) seems to suggest we shall apply our formula with $c_1(x) = -\frac{m(m+1)}{x^2}$ and $c_2(y) = k^2$. Thus, we must find the Green's function G_1 for

$$u_{xx} - u_{yy} - \frac{m(m+1)}{x^2}u = 0 \quad . \tag{4.3}$$

As we have seen previously, G_2 must satisfy (3.25); so

$$G_2(x-X,y;0,Y) = \frac{1}{4i} H_0^{(1)}(kr)$$

where $r \equiv \sqrt{(x-X)^2+(y-Y)^2}$.

In order to avoid an unnecessary digression, we shall show in Appendix B that

$$G_1(x,y-Y;X,0) = \begin{cases} -\frac{1}{2} P_m(\xi) & \text{for } |x-X|<y-Y<x+X \\ 0 & \text{elsewhere} \end{cases} \tag{4.4}$$

where $\xi \equiv \frac{x^2+X^2-(y-Y)^2}{2xX}$ and P_m is the Legendre polynomial of the first kind (see Fig. 2).

Now that G_1 and G_2 are known, we can return to our general form of the addition formula, Eq. (3.19). Then (4.4) may also be written in the form

$$G_1(x,t;X,0) = -\frac{1}{2} P_m\left(\frac{x^2+X^2-t^2}{2xX}\right) H(t-|x-X|)\ H(x+X-t).$$

Noting that $P_m(\xi)$ is the Riemann function $R_1(x,y;X,Y)$ for (4.3), we can also write

$$G_1(x,t;X,0) = -\frac{1}{2} R_1(x,t;X,0)\ H(t-|x-X|)\ H(x+X-t). \tag{4.5}$$

Differentiating with respect to t

$$\begin{aligned} G_{1_t}(x,t;X,0) = &-\frac{1}{2} R_{1_t}(x,t;X,0) H(t-|x-X|) H(x+X-t) \\ &-\frac{1}{2} R_1(x,t;X,0) \delta(t-|x-X|) H(x+X-t) \\ &+\frac{1}{2} R_1(x,t;X,0) H(t-|x-X|) \delta(x+X-t). \end{aligned}$$

Substitution into (3.19) yields

$$G = -2\int_0^\infty G_{1_t} G_2 dt = \int_{|x-X|}^{x+X} R_{1_t} G_2 dt$$

$$+ R_1\Big|_{y-Y=|x-X|} H(x+X-|x-X|)G_2\Big|_{y-Y=|x-X|}$$

$$- R_1\Big|_{y-Y=x+X} H(x+X-|x-X|)G_2\Big|_{y-Y=x+X} . \quad (4.6)$$

Since x and X are both positive, the Heaviside functions are both 1. Since R_1 is the Riemann function for (4.3), $R_1 = 1$ along the characteristic line $y-Y = x-X$; a fact we could also have found by noting that $P_m\Big|_{y-Y=\ x-X} = P_m(1) = 1$. Although we also know

$P_m\Big|_{y-Y=x+X} = P_m(-1) = (-1)^m$, we shall not use this fact

at this stage of the analysis because it appears to be too specialized (i.e., cannot be deduced from properties of the Riemann function) to include in a general form of the formula. Now, since the integrations over the discontinuities of G_1 have been taken into account, we can replace R_1 by $-2G_1$ and using the now familiar argument that G_2 is an even function of t and that the integrand in (4.6) is odd, we get

$$G(x,y;X,Y) = -2\int_{x-X}^{x+X} G_{1_t}(x,t;X,0)G_2(t,y;0,Y)dt$$

$$+ G_2(x-X,y;0,Y)$$

$$+ 2G_1(x,x+X;X,0)G_2(x+X,y;0,Y). \quad (4.7)$$

Equation (4.7) is the counterpart of (3.22). As a matter of fact, in this case we did not need to go all the way back to (3.19). We could have substituted our expression for G_{1_t}, (4.5), into (3.22) and gotten (4.7). However, because we have not proven a theorem as to the applicability of the formula, we gain more confidence by reanalyzing (3.19) than by performing blind substitutions. Now, integrating (4.7) by parts yields the counterpart of (3.23)

$$G(x,y;X,Y) = 2\int_{x-X}^{x+X} G_1(x,t;X,0)G_{2_t}(t,y;0,Y)dt. \quad (4.8)$$

Direct differentiation of (4.7) and (4.8), in a manner analogous to that performed in section 3c, verifies that G is indeed a solution of (4.1) for $(x,y) \neq (X,Y)$. Let us note that because we have finite limits of integration, we no longer need conditions (3.34) and (3.35).

Substituting our known forms of G_1 and G_2 into (4.7) and (4.8), with the added notational convenience $\tilde{r} \equiv \sqrt{(x+X)^2+(y-Y)^2}$, yields

$$G(x,y;X,Y) = \frac{1}{4i}\left\{H_0^{(1)}(kr) - (-1)^m H_0^{(1)}(k\tilde{r}) + \int_{x-X}^{x+X} \frac{\partial P_m}{\partial t}\left(\frac{x^2+X^2-t^2}{2xX}\right)H_0^{(1)}\left(k\sqrt{t^2+(y-Y)^2}\right)dt\right\} \quad (4.9)$$

and

$$G(x,y;X,Y) = \frac{-1}{4i}\int_{x-X}^{x+X} P_m\left(\frac{x^2+X^2-t^2}{2xX}\right) H^{(1)}_{0_t}\left(k\sqrt{t^2+(y-Y)^2}\right)dt. \tag{4.10}$$

So, for any positive integer m, we have shown that the addition formula yields the convergent integrals (4.9) and (4.10) which also satisfy (4.1) for $(x,y) \neq (X,Y)$. If we now differentiate the $H_0^{(1)}$ term as indicated in (4.10) and then perform the change of variables

$$s = \sqrt{t^2+(y-Y)^2}$$

we get

$$G(x,y;X,Y) = \frac{1}{4i}\int_{r}^{\tilde{r}} P_m\left(\frac{x^2+X^2+(y-Y)^2-s^2}{2xX}\right) H_1^{(1)}(ks)k\,ds. \tag{4.11}$$

In this form, we shall show that G has the appropriate singularity, $G \to 0$ as $x \to 0$, and $G \to 0$ as $r \to \infty$.

b. Verification of the Singularity of the Green's Function

For notational convenience in investigating the behavior of G near the singular point $r = 0$, let us write

$$\zeta = \frac{x^2+X^2+(y-Y)^2-s^2}{2xX} \equiv \frac{\tilde{r}^2+r^2-2s^2}{\tilde{r}^2-r^2}\ , \tag{4.12}$$

the identity following directly from the definitions of r and $\tilde{r}$. From (4.12) it is evident that, for

$0 < r \leq s$ and for $s \to 0$, $\zeta \to 1$. Therefore

$$\lim_{0<r\leq s\to 0} P_m(\zeta) = P_m(1) = 1$$

by the continuity of all the functions involved. So given $\varepsilon > 0$ arbitrarily small, choose $\delta > 0$ such that

$$0 < r \leq s \leq \delta \implies 1-\varepsilon \leq P_m(\zeta) < 1 \qquad (4.13)$$

and such that

$$Y_1(ks) < 0$$

is simultaneously satisfied, where of course Y_1 is the first order Bessel function of the second kind (i.e., $Y_1 = \mathrm{Im}\{H_1^{(1)}\}$).

Now for $0 < r < \delta$ rewrite (4.11) as

$$G = \frac{1}{4i}\int_r^{\tilde{r}} P_m(\zeta)J_1(ks)kds + \frac{1}{4}\int_r^{\delta} P_m(\zeta)Y_1(ks)kds$$

$$+ \frac{1}{4}\int_{\delta}^{\tilde{r}} P_m(\zeta)Y_1(ks)kds. \qquad (4.14)$$

The first and third integrals in (4.14) are well-behaved and approach constant values as $r \to 0$. Recalling (4.13), we can apply the mean value theorem to the remaining integral resulting in

$$G = \frac{1}{4} P_m(\zeta^*) \ [Y_0(kr) - Y_0(k\delta)] + \text{constant}$$

where ζ^* is evaluated at $s^* \varepsilon (r,\delta)$. We know that for any $s \varepsilon (0,\delta)$, $1-\varepsilon \le P_m \le 1$ so that

$$\frac{1}{4}(1-\varepsilon)Y_0(kr) + \text{constant} \le G \le \frac{1}{4} Y_0(kr) + \text{constant}.$$

Thus, as $r \to 0$, G behaves like $\frac{1}{4} Y_0(kr)$ or equivalently $\frac{1}{2\pi} \log r$ [Ref. 7, pg. 360, 9.8.1], as desired. We have now verified conditions (1), (2), and (3) of section (3c), that is, the integrals in our representation converge, that G possesses the correct (logarithmic) singularity, and that except at the singularity G satisfies the homogeneous partial differential equation. All that remains is an investigation of the boundary conditions.

c. Verification of the Boundary Condition along $x = 0$

Approaching the singular line $x = 0$ in the half-plane $x > 0$, we have $0 < r < \tilde{r}$. In this case, for $r \le s \le \tilde{r}$, as the limits of integration in (4.11) indicate, we observe from (4.12) that $-1 \le \zeta \le 1$ and therefore $|P_m(\zeta)| \le 1$ [Ref. 8]. We can now write

$$\lim_{x\to 0} |G| \le \lim_{x\to 0} \frac{1}{4} \int_r^{\tilde{r}} |P_m(\zeta)| \cdot |H_1^{(1)}(ks)| k ds$$

$$\le \lim_{x\to 0} \frac{1}{4} \int_r^{\tilde{r}} |H_1^{(1)}(ks)| k ds = 0 \quad ,$$

since as $x \to 0$, $r \to \tilde{r}$ and r is strictly positive (as a matter of fact $r \geq X > 0$ on $x = 0$) and, thus, $|H_1^{(1)}(ks)|$ is a continuous bounded function in the interval from r to $\tilde{r}$.

d. Verification of the Condition at Infinity

Lastly, we investigate the behavior of G as $r \to \infty$. Once again, $0 < r < \tilde{r}$ and $|P_m(\zeta)| \leq 1$ so that, from (4.11),

$$|G| \leq \frac{1}{4}\int_r^{\tilde{r}} |H_1^{(1)}(ks)|kds$$

$$\leq \frac{k}{4}|H_1^{(1)}(kr)|(\tilde{r}-r) \quad , \qquad (4.15)$$

since $|H_1^{(1)}(kr)|$ is a monotonically decreasing function. [Ref. 6, pg. 969, 8.478]. From the definitions of r and $\tilde{r}$ we find that

$$0 < \tilde{r}-r = r[(1 + \frac{4xX}{r^2})^{1/2} - 1].$$

For r sufficiently large, simple manipulations show that

$$0 \leq \tilde{r}-r < 2X + 0(\frac{1}{r}) \quad . \qquad (4.16)$$

For r large, it is also known [Ref. 7, pg. 365, 9.2.28] that

$$|H_1^{(1)}(kr)| = 0(r^{-1/2}) \quad . \qquad (4.17)$$

Substituting (4.16) and (4.17) into (4.15) yields

$$|G| \leq 0(r^{-1/2})$$

and, thus, $G \to 0$ as $r \to \infty$. Hence, we have demonstrated that (4.11) satisfies (4.2), (4.2a), and (4.2b).

e. A Closed Form for $m = 1$

For the case $m = 1$, (4.11) becomes easy to integrate, utilizing the facts that $P_1(\zeta) = \zeta$ and

$$\int s^2 H_1^{(1)}(ks)kds = s^2 H_2^{(1)}(ks) = \frac{2s}{k} H_1^{(1)}(ks) - s^2 H_0^{(1)}(ks) \qquad (4.18)$$

[see Ref. 6, pg. 634, 5.52(1) and pg. 967, 8.471].

From the first equality in (4.18) we integrate (4.11) to

$$G(x,y;X,Y) = \frac{1}{4i} \frac{x^2+X^2+(y-Y)^2}{2xX}[H_0^{(1)}(kr)-H_0^{(1)}(k\tilde{r})] + \frac{1}{4i} \frac{1}{2xX}[r^2H_2^{(1)}(kr)-\tilde{r}^2H_2^{(1)}(k\tilde{r})] \qquad (4.19)$$

$$= \frac{1}{4i} \frac{\tilde{r}^2+r^2}{\tilde{r}^2-r^2} [H_0^{(1)}(kr)-H_0^{(1)}(k\tilde{r})] + \frac{1}{4i} \frac{2}{\tilde{r}^2-r^2}[r^2H_2^{(1)}(kr)-\tilde{r}^2H_2^{(1)}(k\tilde{r})]. \qquad (4.20)$$

Utilizing the second form of (4.18) yields

$$G(x,y;X,Y) = \frac{1}{4i} H_0^{(1)}(kr) + \frac{1}{4i} H_0^{(1)}(k\tilde{r}) - \frac{i}{k(\tilde{r}^2-r^2)}[rH_1^{(1)}(kr)-\tilde{r}H_1^{(1)}(k\tilde{r})] \; . \qquad (4.21)$$

f. The Case $k = 0$

Using standard ascending series forms for $J_n(z)$ and $Y_n(z)$ [Ref. 7, pg. 360, 9.1.10 and 9.1.11], it is easy to show that

$$H_0^{(1)}(z) = \frac{2i}{\pi} \log z + \frac{2i}{\pi} \left[\frac{\pi}{2i} + \gamma - \log 2\right] + O(z^2 \log z)$$

and

$$H_2^{(1)}(z) = \frac{4}{\pi i} \frac{1}{z^2} + \frac{1}{\pi i} + O(z^2 \log z).$$

Substituting these forms into (4.19) and letting $k \to 0$ yields

$$G(x,y;X,Y) = \frac{1}{2\pi}\left(\frac{x^2+X^2+(y-Y)^2}{2xX}\right) \log r/\tilde{r} + \frac{1}{2\pi} . \qquad (4.22)$$

Thus, (4.22) is the Green's function for

$$u_{xx} + u_{yy} - \frac{2}{x^2} u = 0 \qquad (4.23)$$

satisfying the boundary conditions $G \to 0$ as $x \to 0$ and as $|(x-X,y-Y)| \to \infty$. It is interesting to note that (4.22) displays precisely the same form as a fundamental solution as defined by Garabedian [Ref. 9].

If we allow k to approach 0 through positive values in (4.11), we get

$$G(x,y;X,Y) = -\frac{1}{2\pi} \int_r^{\tilde{r}} P_m\left(\frac{x^2+X^2+(y-Y)^2-s^2}{2xX}\right) \frac{ds}{s} . \qquad (4.24)$$

(4.24) can also be derived directly from the addition formula (4.8) where G_1 still is the Green's function for (4.3), but $G_2 = \frac{1}{2\pi} \log r$ is the Green's function for Laplace's equation (3.36). In the case $m = 1$, (4.24) integrates directly to our solution (4.22). Since P_m is simply a polynomial, a straightforward integration of (4.24) may be performed for all positive integers m, yielding closed form Green's functions for

$$u_{xx} + u_{yy} - \frac{m(m+1)}{x^2} u = 0$$

which satisfy the boundary conditions $G \to 0$ as $x \to 0$ as $r \to \infty$.

5. SUMMARY

In section 4, we applied the addition formula to the equation

$$u_{xx} + u_{yy} + [k^2 - \frac{m(m+1)}{x^2}] u = 0 \qquad (5.1)$$

and verified that the integral expression we derived was indeed a Green's function. For the particular case $m = 1$, we were able to express the solution in a closed form. In addition, we succeeded in letting $k \to 0$ in the general integral form yielding Green's functions for (5.1) with $k = 0$. While no claims have been made about the uniqueness of our solutions, we feel that, for these equations the presentation of solutions, in such relatively simple forms, possessing the correct singularity and homogeneous boundary conditions is, in and of itself, of great interest and importance.

Furthermore, the addition formula derived in section 3 yields a new method of searching for Green's functions for separable elliptic partial differential equations. Of course, some analysis must be performed for each problem encountered, but our successful application of the formula to the equations studied in sections 3 and 4 encourages us to attempt to apply the formula to other cases.

REFERENCES

1. E. T. Copson, "On the Riemann-Green Function," Arch. Rational Mech. Anal., 1, 1958, 324-348.

2. A. G. Mackie, "Green's Function and Riemann's Method," Proc. Edinburgh Math. Soc., 14, 1964-1965, 293-302.

3. J. S. Papadakis and D. H. Wood, "An Addition Formula for Riemann Functions," J. Differential Equations (to appear).

4. A. Sommerfeld, Partial Differential Equations in Physics, (Academic Press, 1949), 188-193.

5. A. Erdelyi and others, Tables of Integral Transforms, Vol. II, (McGraw-Hill, 1954), 358, 19. 4.3.

6. I. S. Gradshteyn and I. W. Ryzhik, Table of Integrals, Series, and Products, (Academic Press, Fourth Edition, 1965).

7. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, (National Bureau of Standards, 10th Printing, 1972, with corrections).

8. K. Rektorys, Survey of Applicable Mathematics, (M.I.T. Press, 1969), 723-724, 4. 3.

9. P. R. Garbedian, Partial Differential Equations, (Wiley, 1964), 136-139.

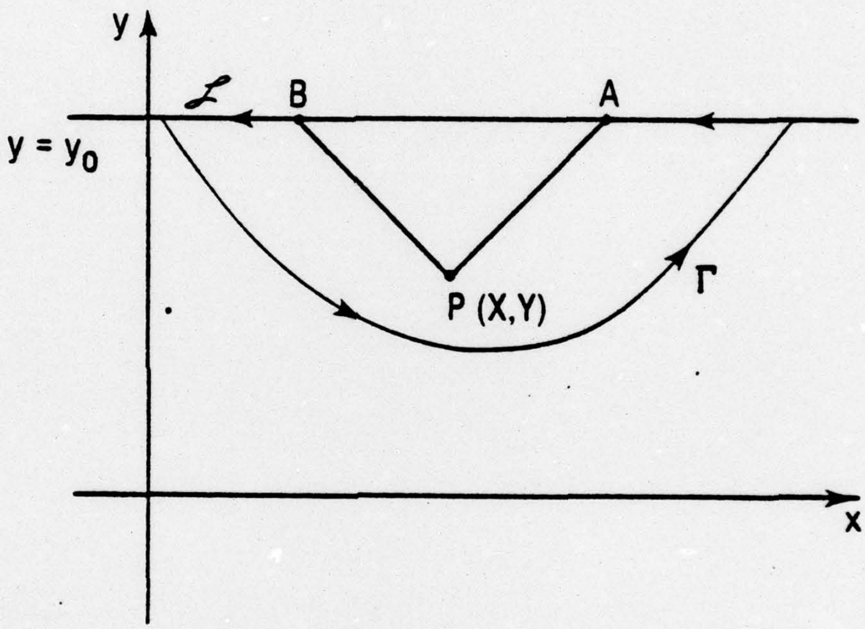


Figure 1: The Green's Function for Hyperbolic Initial Value Problems

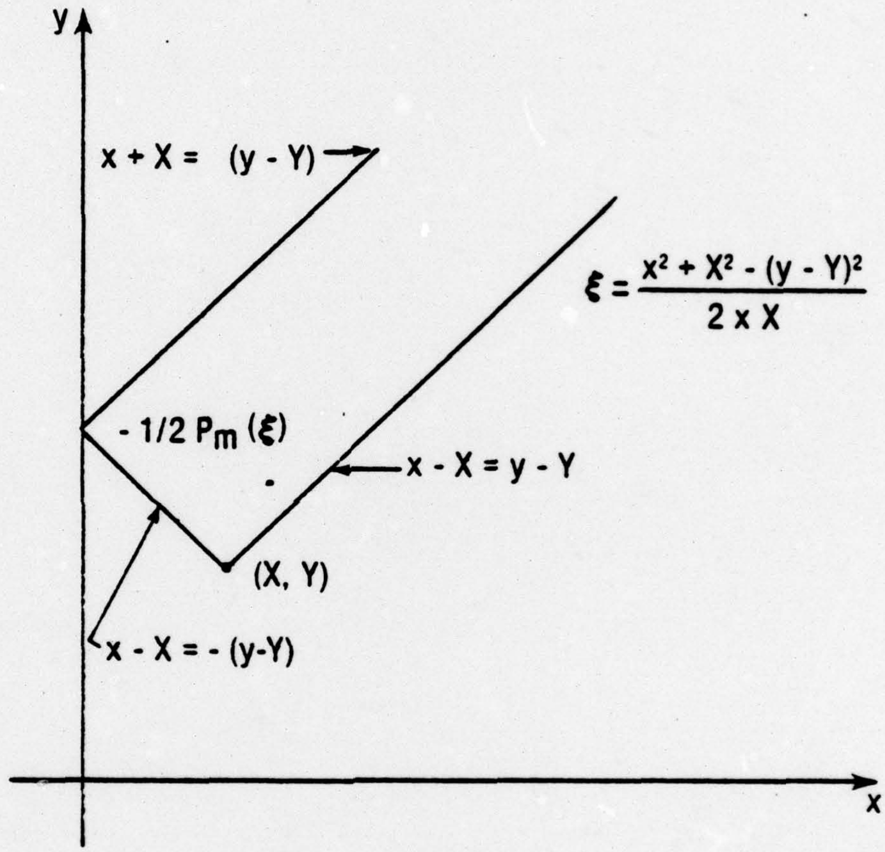


Figure 2: The Green's Function for

$$u_{xx} - u_{yy} - \frac{m(m+1)}{x^2} = 0$$

APPENDICES

Appendix A: OVERVIEW OF THE THESIS

1. INTRODUCTION

The major point of the thesis is the presentation of a technique which yields formulae for Green's functions of linear separable elliptic partial differential equations in two independent variables. The formulae require knowledge of the Green's function (or Riemann function) of a hyperbolic equation and the Green's function of an elliptic equation. Both the hyperbolic and elliptic equations utilized in the process are substantially simpler than the original equation. Thus, known Riemann and Green's functions may be used to generate formulae for new Green's functions. Once a formula for the Green's function is known, one can usually justify the result rigorously using standard analytical techniques. In addition, the overall process is applied in the thesis to find specific Green's functions satisfying homogeneous boundary conditions which we believe to be new solutions.

2. CONTEXT OF THE THESIS

The manuscript deals with the self-adjoint equation

$$u_{xx} + u_{yy} + [c_1(x) + c_2(y)]u = 0. \tag{A.1}$$

In order to find the Green's function $G(x,y;X,Y)$ for (A.1), we investigate the Green's function $G_1(x,t;X,0)$ of the hyperbolic equation

$$u_{xx} - u_{tt} + c_1(x)u = 0 \tag{A.2}$$

and the Green's function $G_2(t,y;0,Y)$ of the elliptic equation

$$u_{tt} + u_{yy} + c_2(y)u = 0. \tag{A.3}$$

The motivation for calling our result an addition formula comes from the fact that formally adding (A.2) and (A.3) yields (A.1).

Employing separation of variables, transform techniques for solving ordinary differential equations, and Parseval's identity from Fourier transform theory, we derive the addition formula

$$G(x,y;X,Y) = G_2(x-X,y;0,Y) - 2\int_{x-X}^{\infty} G_{1_t}(x,t;X,0)G_2(t,y;0,Y)\,dt. \tag{A.4}$$

Integrating by parts and assuming that the term G_1G_2 evaluated at $t = \infty$ is zero, yields the alternative form:

$$G(x,y;X,Y) = 2\int_{x-X}^{\infty} G_1(x,t;X,0)G_{2_t}(t,y;0,Y)dt. \qquad (A.5)$$

The significance of formulas (A.4) and (A.5) is apparent. If $c_1(x)$ is such that the Riemann function of (A.2) is known or can be found and $c_2(y)$ is such that the Green's function of (A.3) is known or can be found, one can substitute them into (A.4) and (A.5) and then directly verify that G satisfies the requirements of a Green's function. As an example, we let $c_1(x) = -\frac{m(m+1)}{x^2}$, so that G_1 is known to be an m-th degree Legendre polynomial of the first kind (see Appendix B for the precise form) and $c_2(y) = k^2$, so that G_2 is a 0-th order Hankel function of the first kind, $H_0^{(1)}(kr)$. Substituting these into (A.5) we get an expression (4.11 in the manuscript) for the Green's function $G(x,y;X,Y)$ of the equation

$$u_{xx} + u_{yy} + [k^2 - \frac{m(m+1)}{x^2}]\, u = 0. \qquad (A.6)$$

In this form, we <u>prove</u> that G is a solution having the correct (logarithmic) singularity and satisfying homogeneous conditions along the singular line $x = 0$ and vanishing at infinity. We believe this result to be new, at least in the particularly simple forms presented in the manuscript. In addition, we perform the necessary integrations and get closed forms for the Green's function of (A.6) for the case $m = 1$ and k arbitrary and for the case $k = 0$ and m arbitrary.

At this point in the thesis development, it was felt that the results should be presented in the open literature for the use of other mathematicians working in this field. However, two relatively straightforward questions are ignored in the manuscript. These deficiencies are eliminated in Appendices C and D. In Appendix C the results of the manuscript are generalized from self-adjoint equations in the form (A.1) to non-self-adjoint equations in the form

$$u_{xx} + u_{yy} + 2a(x)u_x + 2b(y)u_y + [c_1(x)+c_2(y)]u=0 \quad \text{(A.7)}$$

and some non-self-adjoint problems are solved. In Appendix D we solve a problem in which $c_2(y)$ is no longer constant. As a matter of fact we use the results of the manuscript to find the Green's function for

$$u_{xx} + u_{yy} - [\frac{m(m+1)}{x^2} + \frac{n(n+1)}{y^2}]\, u = 0 \qquad \text{(A.8)}$$

in the first quadrant of the plane satisfying homogeneous conditions along the coordinate axes. Once again, this is believed to be a new result. The fact that we could solve otherwise intractable problems like these (i.e., Eqs. (A.6) and (A.8)) indicate the power and utility of the addition formula.

3. SPECULATIVE DISCUSSION

Having derived an addition formula for Green's functions of elliptic equations, one begins to wonder how the result can be generalized. In terms of functional analysis, the partial differential equation is a linear operator and the process of finding the Green's function basically consists of inverting the operator. How can we characterize those linear operators which admit an addition formula solution (to itself or to the inverse problem)? At the present stage of development we cannot answer this question. In order to get a hint towards a direction to proceed, we would like to find other operators for which addition formulae may be found. An addition formula for Riemann functions of linear separable hyperbolic equations in two independent variables has been found by Papadakis and Wood [Reference 3] and their results are stated in Appendix E. However, we still do not have enough examples to guide us.

In section 3c of the manuscript, it is proven that the addition formulas (A.4) and (A.5) are indeed solutions of (A.1) away from the singularity. Following this analysis, it would appear that if, instead of using the Green's function G_2 of (A.3), we used any solution to the homogeneous equation, we get a solution to the homogeneous equation (A.4). Thus, we might be able to develop an addition formula for solutions rather than for Green's functions. Now, totally entering the realm of speculation, it is felt that an addition formula could be derived for elliptic equations in 3 dimensions in terms of a hyperbolic and elliptic decomposition, or possibly in terms of a decomposition consisting of two parabolic equations. It is also guessed that certain types of non-separable equations might admit some form of an addition formula. Further research into these specific areas and into functional analysis techniques (such as those utilized in references 4 and 5) is certainly an interesting avenue of study..

Appendix B: THE GREEN'S FUNCTION FOR

$$u_{xx} - u_{yy} - \frac{m(m+1)}{x^2} u = 0$$

Consider

$$L[G] = G_{xx} - G_{yy} - \frac{m(m+1)}{x^2} G = \delta(x-X)\ \delta(y-Y). \tag{B.1}$$

Multiply both sides of (B.1) by $x^{1/2}J_{m+1/2}(\lambda x)$ and integrate with respect to x from 0 to ∞. After some manipulation (including two integrations by parts and an application of Bessel's ordinary differential equation)

$$g_{yy} + \lambda^2 g = -X^{1/2}J_{m+1/2}(\lambda X)\ \delta(y-Y) \tag{B.2}$$

where

$$g(y;X,Y,\lambda) = \int_0^\infty x^{1/2}J_{m+1/2}(\lambda x)G(x,y;X,Y)\ dx. \tag{B.3}$$

Recalling our previous discussions on the behavior of Green's functions for hyperbolic equations and the similar form of (3.11), the solution of (B.2) is

$$g(y-Y;X,\lambda) = -X^{1/2}J_{m+1/2}(\lambda X)H(y-Y)\frac{\sin\lambda(y-Y)}{\lambda}. \tag{B.4}$$

Noting from (B.3) that g is the Fourier-Bessel transform of $x^{-1/2}G$, we can inverse transform (B.4) and multiply by $x^{1/2}$ to yield

$$G(x,y;X,Y)$$

$$= -(xX)^{1/2}H(y-Y)\int_0^\infty \sin\lambda(y-Y)J_{m+\frac{1}{2}}(\lambda x)J_{m+\frac{1}{2}}(\lambda X)\ d\lambda.$$

Reference to an appropriate table of integrals [Ref. 6, pg. 732, 6.672 (1)] yields the desired result, which is Eq. (4.4) for m a positive integer. For m not an integer,

$$G(x,y;X,Y) = \begin{cases} 0 & \text{for } y-Y<|x-X| \\ -\frac{1}{2} P_m(\xi) & \text{for } |x-X|\le y-Y\le x+X \\ -\frac{\sin m\pi}{\pi} Q_m(-\xi) & \text{for } x+X<y-Y \end{cases}$$

where P_m and Q_m are Legendre functions of the first and second kind, respectively, and $\xi = \dfrac{x^2+X^2-(y-Y)^2}{2xX}$.

Appendix C: NON-SELF-ADJOINT EQUATIONS

Although the manuscript deals only with linear elliptic partial differential equations which are separable and self-adjoint, the non-self-adjoint case may be reduced to self-adjoint form by a transformation of the dependent variable. Let us consider the more general problem

$$L[G] = G_{xx}+G_{yy}+2a(x)G_x+2b(y)G_y+[c_1(x)+c_2(y)]G(x,y;X,Y)$$

$$= \delta(x-X)\ \delta(y-Y). \tag{C.1}$$

Let $\tilde{G}(x,y;X,Y)$ be defined by

$$G(x,y;X,Y) = \exp\{-\int_X^x a(t)dt-\int_Y^y b(t)dt\}\tilde{G}(x,y;X,Y). \tag{C.2}$$

Under this transformation of variables, L defines the transformed operator $\tilde{L}$ by

$$L[G] = \exp\{-\int_X^x a(t)dt - \int_Y^y b(t)dt\}\tilde{L}[G] \tag{C.3}$$

where

$$\tilde{L}[\tilde{G}] = \tilde{G}_{xx} + \tilde{G}_{yy} + [\tilde{c}_1(x) + \tilde{c}_2(y)]\tilde{G}. \tag{C.4}$$

Eq. (C.4) is precisely in the form dealt with in the manuscript. Now if $\tilde{G}$ satisfies

$$\tilde{L}[\tilde{G}] = \delta(x-X)\ \delta(y-Y) \tag{C.5}$$

then Eq. (C.1) is satisfied since the exponential term in Eq. (C.3) becomes unity when $(x,y) = (X,Y)$.

Returning to Eq. (C.4) we note that

$$\tilde{c}_1(x) = c_1(x) - [a(x)]^2 - a'(x), \tag{C.6a}$$

and

$$\tilde{c}_2(y) = c_2(y) - [b(y)]^2 - b'(y). \tag{C.6b}$$

Let $\tilde{L}$ have the form of the example in section 4 of the manuscript, that is

$$\tilde{L}[\tilde{G}] = \tilde{G}_{xx} + \tilde{G}_{yy} + [k^2 - \frac{m(m+1)}{x^2}]\tilde{G} \quad , \tag{C.7}$$

Eq. (4.11) of the manuscript gives us the expression for the Green's function $\tilde{G}$ satisfying $\tilde{G} = 0$ on $x = 0$ and $\tilde{G} \to 0$ as $x^2+y^2 \to \infty$.

Hence we can generate Green's functions for the following non-self-adjoint equations:

$$L_1[G_1] = G_{1_{xx}} + G_{1_{yy}} - \frac{2m}{x} G_{1_x} + k^2 G_1, \tag{C.8}$$

and

$$L_2[G_2] = G_{2_{xx}} + G_{2_{yy}} - \frac{2m}{x} G_{2_x} + kG_{2_y}. \tag{C.9}$$

The Green's function for (C.8) is

$$G_1(x,y;X,Y) = (\frac{x}{X})^m \tilde{G}(x,y;X,Y) \tag{C.10}$$

and for (C.9) is

$$G_2(x,y;X,Y) = (\frac{x}{X})^m e^{-k(y-Y)}\tilde{G}(x,y;X,Y) \ . \tag{C.11}$$

G_1 and G_2 are both 0 on the singular line $x = 0$, but they no longer vanish at infinity.

Appendix D: THE GREEN'S FUNCTION FOR

$$u_{xx} + u_{yy} - [\frac{m(m+1)}{x^2} + \frac{n(n+1)}{y^2}]u = 0.$$

In section 3 of the manuscript, the addition formula was derived for equations of the form

$$u_{xx} + u_{yy} + [c_1(x)+c_2(y)]u = 0.$$

However, in all the examples in the manuscript, c_2 was identically constant. In this appendix we shall find the Green's function $G(x,y;X,Y)$ defined in the first quadrant of the x,y plane $(x > 0, y > 0)$ satisfying

$$G_{xx} + G_{yy} - [\frac{m(m+1)}{x^2} + \frac{n(n+1)}{y^2}]G = \delta(x-X)\ \delta(y-Y) \qquad (D.1)$$

and subject to the boundary conditions

$$G = 0 \text{ on } x = 0 \text{ and on } y = 0.$$

Following the analysis of section 4 of the manuscript, we consider the Green's function $G_1(x,t;X,0)$ of the hyperbolic equation

$$u_{xx} - u_{tt} - \frac{m(m+1)}{x^2}u = 0 \qquad (D.2)$$

which we found to be (see Appendix B)

$$G_1(x,t;X,0) = -\frac{1}{2}P_m(\frac{x^2+X^2-t^2}{2xX})$$

$$\text{for } |x-X| < t < x+X \qquad (D.3)$$

and, for m a positive integer, 0 elsewhere.

Next we need the Green's function $G_2(t,y;0,Y)$ for the elliptic equation

$$u_{tt} + u_{yy} - \frac{n(n+1)}{y^2} n = 0. \tag{D.4}$$

The solution to this problem with boundary condition

$$G_2 = 0 \quad \text{on} \quad y = 0 \tag{D.5}$$

was the subject of section 4f of the manuscript and with a change of notation is precisely Eq. (4.24):

$$G_2(t,y;0,Y) = -\frac{1}{2\pi} \int_{\rho(t)}^{\tilde{\rho}(t)} P_n\left(\frac{y^2+Y^2+t^2-s^2}{2yY}\right)\frac{ds}{s} \tag{D.6}$$

where

$$\rho(t) = \sqrt{t^2+(y-Y)^2} \tag{D.7a}$$

and

$$\tilde{\rho}(t) = \sqrt{t^2+(y+Y)^2}\ . \tag{D.7b}$$

Continuing to follow the analysis of section 4 we find our two forms for G:

$$G(x,y;X,Y) = G_2(x-X,y;0,Y) + 2G_1(x,x+X;X,0)G_2(x+X,y;0,Y)$$

$$-2\int_{x-X}^{x+X} G_{1_t}(x,t;X,0)G_2(t,y;0,Y)dt \tag{D.8}$$

$$=2\int_{x-X}^{x+X} G_1(x,t;X,0)G_{2_t}(t,y;0,Y)dt. \tag{D.9}$$

These forms are identical to Eqs. (4.7) and (4.8) of the manuscript. In Eq. (D.8) the required logarithmic singularity of G is apparent in the first term $G_2(x-X,y;0,Y)$. As a matter of fact, it was proven in the manuscript (section 4b) in an even more general form. Combining (D.5) with (D.8) we find that

$$G(x,0;X,Y) \equiv 0$$

as desired. Also, since G_1 and G_2 are even functions of t, the integrand of (D.9) is odd and from (D.9) we see that

$$G(0,y;X,Y) \equiv 0.$$

Finally, direct differentiation of (D.8) and (D.9) verifies that G is indeed a solution of the homogeneous form of Eq. (D.1) when $(x,y) \neq (X,Y)$.

Substitution of (D.3) and (D.6) into (D.8) yields

$$\begin{aligned} G(x,y;X,Y) = &- \frac{1}{2\pi}\int_{\rho(x-X)}^{\tilde{\rho}(x-X)} P_n\left(\frac{y^2+Y^2+(x-X)^2-s^2}{2yY}\right)\frac{ds}{s} \\ &+ \frac{(-1)^m}{2\pi}\int_{\rho(x+X)}^{\tilde{\rho}(x+X)} P_n\left(\frac{y^2+Y^2+(x+X)^2-s^2}{2yY}\right)\frac{ds}{s} \\ &-2\int_{x-X}^{x+X} \frac{t}{2xX} P_m'\left(\frac{x^2+X^2-t^2}{2xX}\right) \cdot \\ &\cdot \frac{1}{2\pi}\int_{\rho(t)}^{\tilde{\rho}(t)} P_n\left(\frac{y^2+Y^2+t^2-s^2}{2yY}\right)\frac{ds}{s} \cdot dt \end{aligned} \qquad (D.10)$$

where ρ and $\tilde{\rho}$ are as defined in (D.7a) and (D.7b). The situation simplifies somewhat in the case $m=1=n$. Then

$$G(x,y;X,Y) = \frac{1}{2\pi} + \frac{1}{2\pi}\,\frac{y^2+Y^2+(x-X)^2}{2yY}\,\log\frac{\rho(x-X)}{\tilde{\rho}(x-X)}$$

$$+ \frac{1}{2\pi} + \frac{1}{2\pi}\,\frac{y^2+Y^2+(x+X)^2}{2yY}\,\log\frac{\rho(x+X)}{\tilde{\rho}(x+X)}$$

$$-2\int_{x-X}^{x+X}\frac{t}{2xX}\left\{\frac{1}{2\pi} + \frac{1}{2\pi}\,\frac{y^2+Y^2+t^2}{2yY}\log\frac{\rho(t)}{\tilde{\rho}(t)}\right\}dt. \quad \text{(D.11)}$$

The integral may be written in a closed form consisting of expressions containing powers of $(x \pm X)$ and $(y \pm Y)$ and logarithms of $(x \pm X)^2 + (y \pm Y)^2$ [see reference 6, pg. 205, 2.732].

Appendix E: AN ADDITION FORMULA FOR RIEMANN FUNCTIONS FOR HYPERBOLIC EQUATIONS

As mentioned in section 1 of the manuscript, the motivation for this thesis in elliptic partial differential equations came from recent advances in the theory of Riemann functions for hyperbolic partial differential equations. Because of the nice properties of hyperbolic equations (e.g., the Riemann function is the solution to a homogeneous equation and, thus, has no singularity, the domain of influence of hyperbolic equations is bounded by two real characteristics and thus finite in the spatial direction, the Riemann function does not depend on boundary or initial conditions) it is much simpler to state and prove an addition formula in the hyperbolic case. The following theorem and corollary may be found in reference 3. The proof consists basically of a change of dependent variable to transform the equation from non-self-adjoint form into self-adjoint form (as was done in Appendix C of this thesis) and direct substitution and differentiation in the equation (as was done in section 3c of the manuscript).

THEOREM. If $R_1(x,y,X,Y)$ and $R_2(x,y,X,Y)$ are the respective Riemann functions for

$$U_{xx} - U_{yy} + 2b_1(x)U_x + c_1(x)U = 0$$

and

$$U_{xx} - U_{yy} - 2b_2(y)U_y - c_2(y)U = 0,$$

then the Riemann function $R(x,y,X,Y)$ for

$$U_{xx}-U_{yy} + 2b_1(x)U_x - 2b_2(y)U_y + [c_1(x)-c_2(y)]U = 0$$

is given either by

$$R(x,y,X,Y) = R_1(x,y-Y,X,0)\ \exp[\int_Y^y b_2(t)dt] + \int_{y-Y}^{x-X} R_1(x,t,X,0)\ R_{2_t}(t,y,0,Y)\ dt$$

or by

$$R(x,y,X,Y) = R_2(x-X,y,0,Y)\ \exp[\int_X^x b_1(t)\ dt] + \int_{x-X}^{y-Y} R_2(t,y,0,Y)\ R_{1_t}(x,t,X,0)\ dt.$$

COROLLARY. Consider the equation

$$U_{rs} + 4b_1(r+s)[U_r+U_s] + 4b_2(r-s)[U_r-U_s] + 4[c_1(r+s) - c_2(r-s)]U = 0. \tag{E.1}$$

If the Riemann function of (E.1) is $V_1(r,s,R,S)$ when $b_2 = c_2 = 0$ and is $V_2(r,s,R,S)$ when $b_1 = c_1 = 0$, then the Riemann function of (E.1) is given by

$$V(r,s,R,S) = V_1(r,s,R,S) \exp \left[\int_{R+S}^{r+s} b_1(t)\,dt\right]$$

$$+ \int_{r-R-s+S}^{r-R+s-S} V_1\left(\frac{r+s+t}{2}, \frac{r+s-t}{2}, \frac{R+S}{2}, \frac{R+S}{2}\right) \cdot$$

$$\cdot\, V_{2_t}\left(\frac{t+r-s}{2}, \frac{t-r+s}{2}, \frac{R-S}{2}, \frac{-R+S}{2}\right) dt \,.$$

Appendix F: BIBLIOGRAPHY

1. E. T. Copson, "On the Riemann-Green Function," *Arch. Rational Mech. Anal.*, 1, 1958, 324-348.

2. A. G. Mackie, "Green's Function and Riemann's Method," *Proc. Edinburgh Math. Soc.*, 14, 1964-1965, 293-302.

3. J. S. Papadakis and D. H. Wood, "An Addition Formula for Riemann Functions," *J. Differential Equations* (to appear).

4. B. Friedman, "An Abstract Formulation of the Method of Separation of Variables," New York University, Report No. BR-12, 1955.

5. F. J. Murray and J. V. Neumann, "On Rings of Operators," *Annals of Math.* 37 (1936), 116-229.

6. A. Sommerfeld, *Partial Differential Equations in Physics*, (Academic Press, 1949), 188-193.

7. A. Erdelyi and others, *Tables of Integral Transforms*, Vol. II, (McGraw-Hill, 1954), 358, 19. 4.3.

8. I. S. Gradshteyn and I. W. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, Fourth Edition, 1965).

9. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (National Bureau of Standards, 10th Printing, 1972, with corrections).

10. K. Rektorys, *Survey of Applicable Mathematics*, (M.I.T. Press, 1969, 732-724, 4.3.)

11. P. R. Garbedian, *Partial Differential Equations*, (Wiley, 1964), 136-139.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
ASN(R&D)	1
ONR, Code 102-OS, 412-3, 480	4
CNM, MAT-0302, -03R	3
NRL	1
NAVSEA, SEA-03C, -09G3(4)	5
NAVAIRDEVCEN	1
NAVWPNSCEN	1
DTNSRDC	1
NAVCOASTSYSLAB	1
NOSC	1
NAVWARCOL	1
NETC	1
APL/UW, SEATTLE	1
ARL/PENN STATE, STATE COLLEGE	1
DDC, ALEXANDRIA	12
WOODS HOLE OCEANOGRAPHIC INSTITUTION	1
NAVOCEANO, Code 02	2